Phase synchronization of two-dimensional lattices of coupled chaotic maps

Bambi Hu[1,2] and Zonghua Liu[1,3]
[1]Department of Physics and Center for Nonlinear Studies, Hong Kong Baptist University, Hong Kong, China
[2]Department of Physics, University of Houston, Houston, Texas 77204
[3]Department of Physics, Guangxi University, Nanning 530004, China
(Received 3 August 1999)

Phase synchronized states can emerge in the collective behavior of an ensemble of two-dimensional chaotic coupled map lattices, due to a nearest-neighbor interaction. A definition of phase is given for iterated systems, which corresponds to the definition of phase in continuous systems. The transition to phase synchronization is characterized in an ensemble of lattices of logistic maps, in terms of the phase synchronization ratio, the average abnormal ratio, and conditional Lyapunov exponents. The largest Lyapunov exponent of the global system λ_{max} depends on both the number of coupled maps and the coupling strength. If the number of coupled maps is over some threshold, λ_{max} depends only on the coupling strength. The approach of nearest-neighbor coupling is robust against a small difference in the map parameters.

PACS number(s): 05.45.Xt, 47.54.+r

I. INTRODUCTION

The synchronization of chaotic systems has attracted considerable attention in recent years [1–6]. This synchronization has clear applications to communications, control and anticontrol of chaos in biomedical systems, and system identification. Most recently, the concept of chaos synchronization has been extended to that of phase synchronization (PS) of chaotic systems [7]. In this process, the interaction of nonidentical chaotic systems can lead to a perfect locking of their phases, whereas their amplitudes remain uncorrelated. Phase locking of chaotic signals in large populations of coupled dynamical units, where each separate unit may reside on a chaotic attractor, is currently a subject of active investigations [8]. This paper is devoted to this subject. Usually, this PS phenomenon is observed and studied in continuous flows. Little attention is paid to PS in iterated systems. In this paper we report a new phenomenon that the local activity of each map can show PS. We will give a general definition of PS in iterated systems and demonstrate that the short-range (diffusive) interaction can also induce the network to display PS. The mechanism of PS can be quantitatively characterized by appropriate measures, such as the PS ratio, the average abnormal ratio, and conditional Lyapunov exponents, etc.

Spatiotemporal chaos and complex pattern formation has been studied extensively in networks of coupled maps [9,10] among which coupled map lattices are more simply organized objects due to a diffusive coupling. So far, the study of globally coupled dynamical systems has revealed novel concepts [11–13] such as clustering, chaotic itinerancy, and partial ordering. In the present paper, we are interested in the network with diffusive nearest-neighbor coupling. We study the PS of two-dimensional (2D) coupled logistic maps and how the number of coupled elements, coupling strength, and parameter mismatch affect the PS.

This paper is organized as follows. In Sec. II we describe our model of 2D coupled map lattices with nearest-neighbor interaction. In Sec. III we investigate the behavior and mechanism of PS and define appropriate measures to quantitatively characterize the PS. In Sec. IV we study the robustness of PS against a small parameter mismatch. Finally, in Sec. V a brief discussion and summary are given.

II. DESCRIPTION OF THE MODEL

The system we studied consists of $m=N\times N$ units, each of which is a logistic map. The dynamical properties of each map are determined by its parameter $\gamma^{i,j}$, which may be different for each map. The variable of each map $x^{i,j}$ is iterated forward to time step $n+1$ by

$$x_{n+1}^{i,j}=f_{i,j}(\gamma^{i,j},x_n^{i,j})=\gamma^{i,j}x_n^{i,j}(1-x_n^{i,j}), \tag{1}$$

where $x\in[0,1]$ and $\gamma\in[0,4]$. Then for a 2D array consisting of $N\times N$ oscillators with diffusive nearest-neighbor coupling, the equation is

$$x_{n+1}^{i,j}=\frac{1}{1+\alpha}\Big\{f_{i,j}(\gamma^{i,j},x_n^{i,j})+\frac{\alpha}{4}[f_{i,j-1}(\gamma^{i,j-1},x_n^{i,j-1}) + f_{i,j+1}(\gamma^{i,j+1},x_n^{i,j+1})+f_{i-1,j}(\gamma^{i-1,j},x_n^{i-1,j}) + f_{i+1,j}(\gamma^{i+1,j},x_n^{i+1,j})]\Big\},$$

$$\gamma^{i,j}=\gamma_0(1+a\xi_{i,j}), \tag{2}$$

where $i,j=1,2,\ldots,N$, and $\gamma_0\in(3.57,4.0)$. α denotes the strength of nearest-neighbor coupling, a the detuning coefficient of parameter, and $\xi_{i,j}$ a random number between -1 and 1, thus $\gamma^{i,j}$ is different for each unit. We will assume free boundary conditions, i.e., $x^{0,j}=x^{1,j},x^{i,0}=x^{i,1},x^{N+1,j}=x^{N,j}$, and $x^{i,N+1}=x^{i,N}$, dependent on its previous state and the average state of the nearest-neighbor maps. The value of the iteration at the given site is then normalized not to exceed 1. Starting from this model, in the following we will first introduce a definition of PS in iterated systems, then give a quantitative characterization of PS, and finally explain the mechanism of PS.

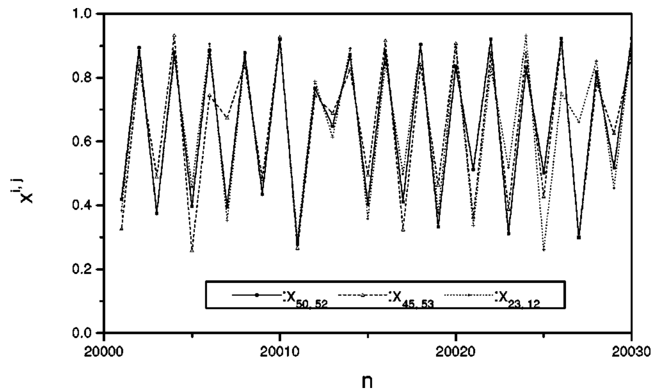

FIG. 1. Temporal evolution of the activities of the units in the lattice for parameters $\gamma_0=3.755$, $a=0.0$, $\alpha=1.0$, and $N=100$. The three displayed signals correspond to three different lattice units $x^{50,52}$, $x^{45,53}$, and $x^{23,12}$, respectively.

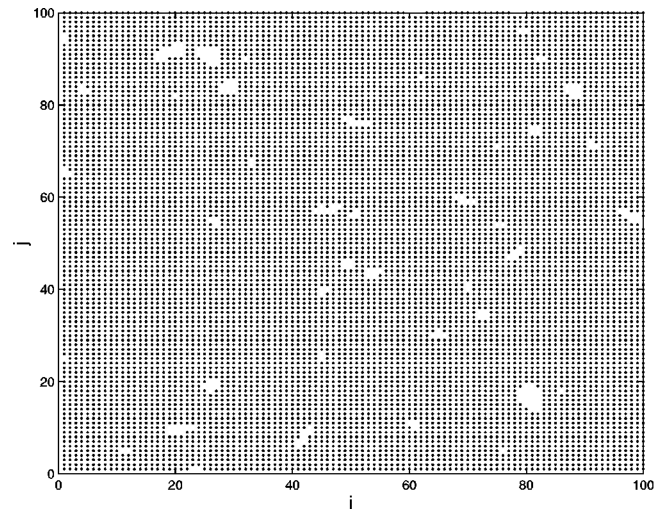

FIG. 2. Phase state after 20 000 iterations for $\gamma_0=3.755$, $a=0.0$, $\alpha=1.0$, and $N=100$, where the black dots corresponding to units having phase 1, and the empty area corresponding to units having phase 0.

III. ANALYSIS OF PS

First, we consider the case of identical units. Equation (2) describes the behavior of $N\times N$ identical coupled logistic maps, all in the chaotic state. In the following, we focus our attention on a system with $N=100$, starting from random initial conditions, and with free boundary conditions. Our numerical simulation shows that with increasing coupling strength α, more and more units have a similar oscillatory behavior. This similarity will not change with time. We call the states that show local maxima (minima) at the same time as phase synchronized states. Figure 1 shows the oscillatory behaviors of three units for the coupling strength $\alpha=1.0$. Obviously, they have maxima (minima) at the same time but different amplitudes. In fact, if we check the amplitudes of all the units at some time, we will find that these amplitudes are chaotic. Now, the key problem is to find a good definition of the phase in an iterated system. In continuous flows, the direction of trajectory corresponds to the zeroth Lyapunov exponent. Usually, the phase is defined as a variable that corresponds to the zeroth Lyapunov exponent of a continuous-time dynamical system that displays chaotic behavior [14]. We think that the phase of iterated systems should have something to do with the direction of iteration. In order to quantitatively characterize the transition to PS in iterated systems, we now define the phase of the (i,j)th lattice $P_n^{i,j}$ at some time n as

$$P_n^{i,j}=\begin{cases}1, & \text{if } x_n^{i,j}/x_{n-1}^{i,j}>1;\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

As the units are always oscillatory, same $P_n^{i,j}$ denote PS or phase cluster. Figure 2 shows the results when $\alpha=1.0$ and $n=20\,000$. From this figure one can see that at this moment most of the units have phase 1 but a small part have phase 0. Now, we define the PS ratio r as

$$r=\frac{1}{N\times N}\max\left\{\sum_{i,j=1}^{N}\,(P_n^{i,j}=1),\sum_{i,j=1}^{N}\,(P_n^{i,j}=0)\right\}, \tag{4}$$

where $\Sigma_{i,j=1}^{N}(P_n^{i,j}=1)$ and $\Sigma_{i,j=1}^{N}(P_n^{i,j}=0)$ denote the number of units in phase 1 and the number of units in phase 0, respectively. If the phases of all the units change alternatively between 0 and 1, this definition guarantees that r remains constant. Two questions may arise: (1) Will r change with time when the coupling strength α is fixed? (2) How does the coupling strength α affect r?

To answer the first question, we investigate the PS ratio r corresponding to different γ_0 in range (3.57,4.0). We find that there are two kinds of different r corresponding to regions of γ_0(3.57,3.677) and (3.678,4.0), respectively. r will not change with time if γ_0 is in (3.57,3.677) and change with time if γ_0 is in (3.678,4.0). Why do they show different properties? If γ_0 is in the region (3.57,3.677), the phases of all the units are regularly changed with time between 1 and 0. The phase series of the (i,j)th lattice, $P_n^{i,j}$ (for $n=1,2,3\ldots$), is 10101010 . . . . We call the chaotic behavior of the amplitude in this region of γ_0 (or whose phase has this property) as simple chaos. While in the region $\gamma_0\in(3.678,4.0)$, the phase does not always change alternatively with time. It is possible for two consecutive iterations to have the same phase, such that e.g., $P_n^{i,j}$ (for $n=1,2,3\ldots$) will be 0101101010 . . . . We call the chaotic behavior of the amplitude in this region of γ_0 complex chaos. To quantitatively characterize the complexity, we define an average abnormal ratio σ:

$$\sigma=\frac{1}{T}\sum_{t=0}^{T}\frac{1}{N\times N}\sum_{i,j}\ \sigma_{i,j}(t) \tag{5}$$

where

$$\sigma_{i,j}(t)=\begin{cases}1, & \text{if } P_n^{i,j}=P_{n-1}^{i,j};\\ 0, & \text{otherwise.}\end{cases}$$

For example, if the phase series of the i,jth lattice reads 0101101010 . . . , $\sigma_{i,j}$ becomes 000100000 . . . . Hence, in iterated systems, the chaotic behavior will be denoted as simple chaos, if $\sigma=0$, and as complex chaos, if $\sigma>0$, respectively. Later, we will use this quantity σ to explain the phenomenon that r changes with the coupling strength α.

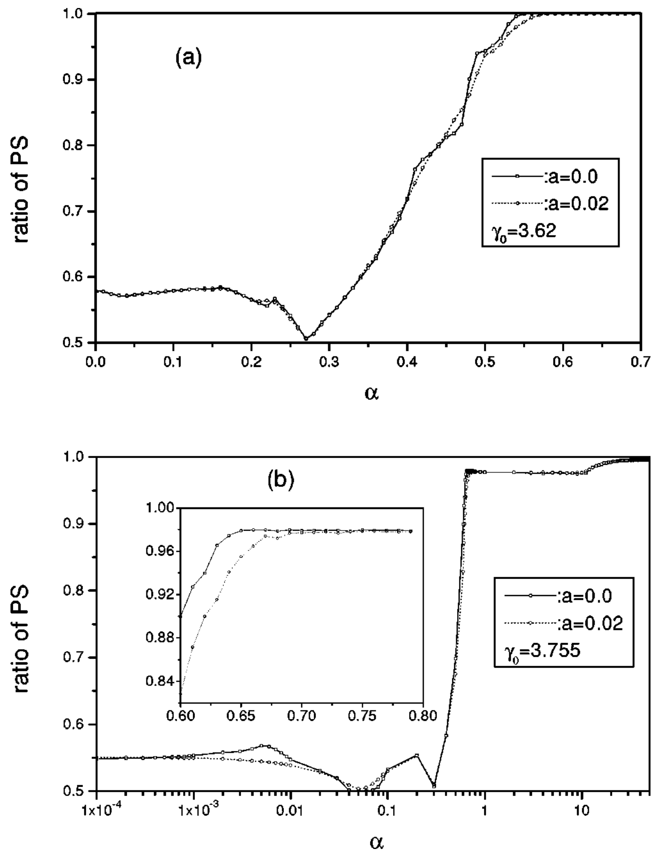

FIG. 3. Relationship between r and coupling strength α for $N=100$. (a) $\gamma_0=3.62$; (b) $\gamma_0=3.755$, inset: local blowup for $\alpha\in[0.6,0.8]$.

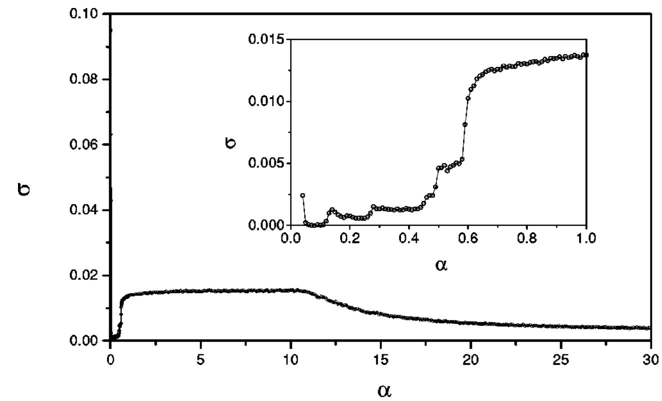

FIG. 4. Relationship between the average abnormal ratio σ and the coupling strength α for $\gamma_0=3.755$. Inset: the local blowup for $\alpha\in[0.04,1.0]$.

To answer the second question, we investigate the relationship between r and the coupling strength α. We find that for simple chaos, r will quickly reach 1.0 with increasing coupling strength α, while for complex chaos, r quickly reaches some value (which is a little smaller than 1.0) and then very slowly approaches 1.0. Figure 3 shows two typical results, where (a) denotes the case of simple chaos and (b) the case of complex chaos. As r depends on time for $\gamma_0=3.755$, the points in Fig. 3 are long-time average. From Fig. 3(a) one can see that r equals 1.0 if $\alpha\geqslant 0.55$. That is, all of the units have become phase synchronized. From Fig. 3(b) one can see that r quickly increases if α is in the interval $[0.3\sim 0.7]$ and then slowly increases with increasing α. The inset of this figure shows the local blowup for $\alpha\in[0.6,0.8]$. Comparing Fig. 3(a) with Fig. 3(b) we can see that the most obvious difference is that r can reach 1.0 in Fig. 3(a) but not in Fig. 3(b). What is the reason? Now we use Eq. (5) to give an explanation. Here we focus on the case of complex chaos with $\sigma>0$. Our numerical experiments show that the average abnormal ratio σ will change with the coupling strength α. Figure 4 gives the relationship between the average abnormal ratio and the coupling strength α when $\gamma_0=3.755$, where the inset shows the local blowup for $\alpha\in[0.04,1.0]$. From Fig. 4 one can see that $\sigma=0.095$ if $\alpha=0$, and then quickly decreases if $\alpha\leqslant 0.05$, followed by some platform if $\alpha\in(0.05,11.0)$, and finally decreases again if $\alpha\geqslant 11.0$. Obviously, the coupling can reduce the average abnormal ratio σ. What is the relationship between α and r? Comparing Fig. 4 with Fig. 3(b) one can see that there is some intrinsic connection between the two figures. When the coupling strength α is small ($\alpha\leqslant 0.4$), it suppresses the average abnormal ratio but is not enough to induce PS. So r is in (0.5, 0.6). When α is near 0.5 or 0.6, there is a jump for σ in Fig. 4. It corresponds to the part of rapidly increasing r in Fig. 3(b). When α is in the range $[0.7,11.0]$, both the σ in Fig. 4 and the r in Fig. 3(b) are approximately constant. If $\alpha>11.0$, σ begins to decrease and r begins to increase further. So the average abnormal ratio σ is tightly connected with r. Because the $\sigma>0$ represents the intrinsic feature of complex chaos, we cannot make the system become the case of simple chaos with $\sigma>0$ by finite coupling. That is the reason why the r is difficult to reach 1.0. For illustrating it in more detail, we investigate its Lyapunov exponent.

For each map the coupling part in Eq. (2) $(\alpha/4)(f_{i,j-1}+f_{i,j+1}+f_{i+1,j}+f_{i-1,j})$ can be treated as an external signal. If we treat the four nearest-neighbor lattices as the driving system and the i,jth lattice as the responding system, according to Pecora and Carroll's method [1], the conditional Lyapunov exponent of response system can be given as

$$\lambda_{con}^{i,j}=\lim_{M\to\infty}\frac{1}{M}\sum_{n=1}\log\left|\frac{1}{1+\alpha}f'_{i,j}(\gamma^{i,j},x_n^{i,j})\right|$$
$$=\lim_{M\to\infty}\frac{1}{M}\sum_{n=1}[\log(\gamma^{i,j}|1-2x_n^{i,j}|)-\log(1+\alpha)]. \tag{6}$$

From Eq. (6) one can see that $\lambda_{con}^{i,j}$ will decrease with increasing α. When $\lambda_{con}^{i,j}<0$, the unit (i,j) will synchronize with its neighbors. Figure 5 shows the results of three typical units. Obviously, when $\alpha>0.2$, they all become negative. A question may arise: why don't we see the appearance of the identical synchronization when $\lambda_{con}^{i,j}<0$? We know that in Pecora and Carroll's method [1] there are only two systems. One is the driving system and the other is the response system. When the largest conditional Lyapunov exponent become negative, the response system will become synchronized with the driving system. However, in our case there are four nearest-neighbor units, and the driving signal is the average of the four nearest-neighbor units. So the response system should go to the state corresponding to the average term.

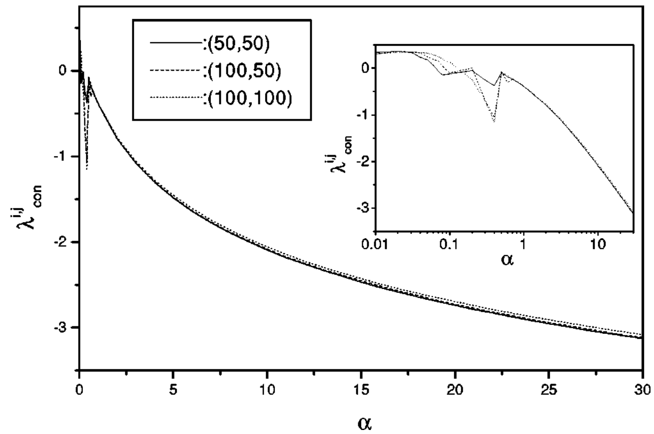

FIG. 5. Conditional Lyapunov exponent for different sites: $\gamma_0 = 3.755$ and $N = 100$. Inset: plot of the conditional Lyapunov exponent versus the logarithm of α. The transition values of α for $\lambda^{i,j}_{con} = 0$ are $\alpha = 0.07$, 0.09, and 0.2, respectively.

As the average is different for each unit, it cannot lead to identical synchronization of all the units. The effect of the average term is to change the running direction of the unit towards a common direction. Hence, the role of the average term is to make the neighboring units have the same phase and lead all the units toward the state of PS. On the other hand, let $\lambda^{i,j}_{con} = 0$ in Eq. (6), one can see that the α is different for different lattices (i,j) because of the boundary effects. From the inset of Fig. 5 one can see that the transition values of α are 0.07, 0.09, and 0.2, respectively. That is, at the transition points of $\lambda^{i,j}_{con} = 0$, the needed coupling strength for boundary units is larger than for inner units.

To characterize how the size of the network affects the global dynamical behavior of Eq. (2), we now compute the largest Lyapunov exponent of the global system λ_{max} for different N. Our numerical experiments reveal that λ_{max} depends on both the size N and the coupling strength α. Figure 6 shows the results. From this figure one can see that λ_{max} will change with increasing α. For $N = 2$, λ_{max} is negative if $\alpha \in [0.2, 0.3]$ or $\alpha \in [0.5, 1.0]$. It means that the mutual interaction between the 2×2 units drive the whole system into periodic behavior. If $\alpha > 2.0$, λ_{max} is approximately equal to the λ_{max} in case of no interaction. It illustrates that the 2×2 coupled units have become identical synchronization. For $N = 4$ and $\alpha > 5.0$, λ_{max} becomes negative. It shows that

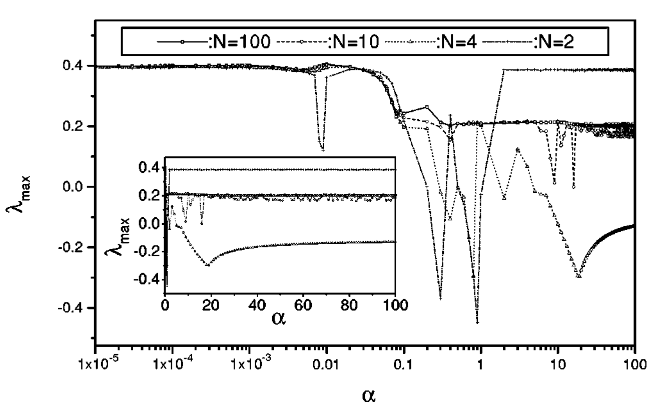

FIG. 6. Largest Lyapunov exponent for different sizes N: $\gamma_0 = 3.755$. Inset: plot of λ_{max} versus α.

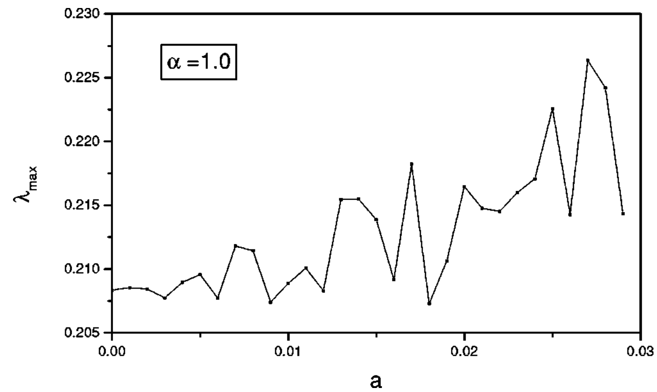

FIG. 7. Relationship between the largest Lyapunov exponent and the parameter mismatch for $\gamma_0 = 3.755$, $\alpha = 1.0$, and $N = 100$.

the whole system displays periodic behavior. For $N = 10$, λ_{max} is positive for all coupling strengths α, but shows large fluctuations if α increases. It illustrates that the global behavior is sensitive to the coupling and the boundary effect is important for this size of the network. For $N = 100$, λ_{max} has two main values with increasing α. λ_{max} is at the higher value if $\alpha \leqslant 0.02$ and the lower value if $\alpha \geqslant 0.4$. The lower value of λ_{max} corresponds to the state of PS. The region of $\alpha \in [0.02, 0.4]$ denotes the transition region in which the phase behavior of the global system changes from an irregular state to the phase-synchronized state. If N increases further, the curve of λ_{max} is just the same as for $N = 100$ except for a smaller fluctuation. So if N is over some threshold, the curve of λ_{max} becomes smooth and does not depend on N. If α is also over some threshold, λ_{max} will keep constant. Hence we conclude that for large N, Eq. (2) can display stabilized PS and the whole system cannot display periodic behavior. For small N, Eq. (2) can show periodic behavior. As one knows, if all the units have become identically synchronized, λ_{max} will not change with increasing coupling. Our simulations show that all the λ_{max} will change with increasing α except for $N = 2$. So only for $N = 2$ all the units can show identical chaotic synchronization. It confirms that the interaction behavior of an ensemble of coupled units is more complex than that of a few units.

IV. ROBUSTNESS

In real systems, we cannot make all the units identical. Here we consider the case that the parameters $\gamma^{i,j}$ have random fluctuations. Every unit runs in a different chaotic region. What is the behavior of the whole coupled system? Our numerical simulation shows that PS can still appear when α is over some threshold. $\sigma = 0$ leads to a more rapid and robust phase locking of the coupled maps than $\sigma > 0$. Figure 3 shows the result of $a = 0.02$. From Figs. 3(a) and 3(b) one can see that $r(\alpha)$ for $a = 0.02$ is approximately the same as for $a = 0.0$ except for some small fluctuations. It illustrates that PS has some robustness against a parameter mismatch.

On the other hand, because of the fluctuation of the parameters, the behavior of the global system will have some changes. We use the largest Lyapunov exponent λ_{max} to illustrate the changes. Figure 7 shows the result for $\gamma_0 = 3.755$ if the coupling strength $\alpha = 1.0$. Obviously, λ_{max} shows fluctuations that become larger and larger with in-

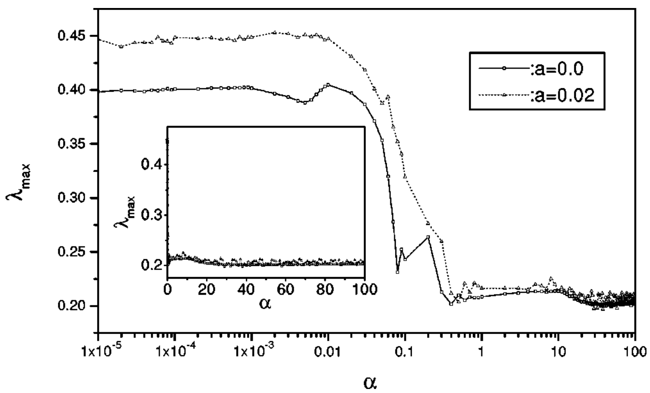

FIG. 8. Relationship between the largest Lyapunov exponent and the coupling strength under different parameter mismatches for $\gamma_0 = 3.755$ and $N = 100$. Inset: plot of λ_{max} versus α.

creasing parameter mismatch a. This can be explained as follows. If a increases, the parameter difference between two units may become larger and the global system becomes more complex. So this increases the difficulty of forming PS, independently on the coupling strength. Figure 8 confirms this result. In Fig. 8, one can see that if $\alpha < 10$, the exponents λ_{max} for $a = 0.02$ are larger than those for $a = 0.0$, and if $\alpha > 10$, the exponents for $a = 0.02$ are fluctuating around those for $a = 0$. Thus, the global behavior of nonidentical units is sensitive to the coupling strength. Comparing Fig. 8 with Fig. 3 one can see that for the two cases of $a = 0.0$ and $a = 0.02$ the difference of λ_{max} in Fig. 8 is large while the difference of r in Fig. 3 is small.

V. DISCUSSION AND CONCLUSIONS

The above results are obtained by free-end conditions, but our numerical experiments show that they are also correct for periodic boundary conditions except for small differences. Additionally, the above results can be extended to 1D coupled maps and other 2D coupled maps. For example, in 2D coupled Hénon maps we observed a similar phenomenon of PS; in 1D coupled logistic maps we observed some phase clusters that have the same phase but different amplitudes. So PS does not only appear in continuous systems, it can also appear in iterated maps as an emergent phenomenon. The deeper study shows that there are some scaling exponents in coupled maps. This will be reported elsewhere.

In summary, we have considered a 2D network of coupled logistic maps. We have shown that a phase synchronized state can emerge in the collective behavior of an ensemble of chaotic coupled map lattices, due to a nearest-neighbor interaction. By introducing a definition of phase in iterated systems, we can quantitatively characterize the features of PS in iterated systems and explain the mechanism of transition to PS by the average abnormal ratio σ and the conditional Lyapunov exponent. We call the case with $\sigma = 0$ as simple chaos and the case with $\sigma > 0$ as complex chaos. In the case of complex chaos, it is difficult to get the complete phase synchronization $(r = 1)$. Furthermore, if the size N of the system is small, the global system displays both periodic and chaotic behavior. When N is large, the global system displays only chaotic behavior. When N is over some threshold, the largest Lyapunov exponent λ_{max} does only depend on the coupling strength. For the case of different parameters, PS can also be implemented except for a little fluctuation. So this method of nearest-neighbor coupling is robust against a small difference in the map parameters.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Wei Wang for fruitful discussions. This work was supported in part by grants from the Hong Kong Research Grant Council, the Hong Kong Baptist University Faculty Research Grant, and the National Natural Foundation of China.

[1] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[2] R. He and P.G. Vaidya, Phys. Rev. A **46**, 7387 (1992).
[3] K. Pyragas, Phys. Lett. A **181**, 203 (1993).
[4] N.F. Rulkov, M.M. Sushchik, L. Tsimring, and H.D.I. Abarabanel, Phys. Rev. E **51**, 980 (1995).
[5] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).
[6] Zonghua Liu and Shigang Chen, Phys. Rev. E **56**, 168 (1997); **55**, 6651 (1997); **56**, 1585 (1997).
[7] M.G. Rosenblum, A.S. Pikovsky and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996); **78**, 4193 (1997).
[8] D.H. Zanette and A.S. Mikhailov, Phys. Lett. A **235**, 135 (1997); D.H. Zanette, Phys. Rev. E **55**, 5315 (1997).
[9] K. Kaneko, Physica D **75**, 55 (1994); **77**, 456 (1994); **86**, 158 (1995).
[10] Zonghua Liu, Shigang Chen, and Bambi Hu, Phys. Rev. E **59**, 2817 (1999).
[11] P. Grassberger, Phys. Rev. E **59**, R2520 (1999).
[12] G. Abramson and D.H. Zanette, Phys. Rev. E **58**, 4454 (1998).
[13] T. Shibata and K. Kaneko, Physica D **124**, 177 (1998).
[14] A.S. Pikovsky, M.G. Rosenblum, G.V. Osipov, and J. Kurths, Physica D **104**, 219 (1997).